# Multistage Blisk Large Mistuning Modeling Using Fourier Constraint Modes and Pristine Rogue Interface Modal Expansion

Undergraduate Honors Thesis

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## Abstract

In the turbomachinery field, a great deal of research focus has been on understanding the effects of mistuning on the dynamics of turbine bladed disks (blisks). Mistuning in blisks is due to the blade to blade differences that typically occur from manufacturing tolerances and in-service wear. Mistuning greatly complicates the modeling of the blisk since it is inherently random and requires a statistical analysis to be conducted to understand the full dynamics of a bladed disk design. Moreover, mistuning destroys the cyclic symmetry of the bladed disk, which prevents cyclic analysis from being used to model the system. Cyclic analysis enables single sector models and calculations to be employed to analyze the full stage dynamics. A wide array of methods to efficiently model (using single sector models and calculations)the mistuning in these structures have been developed to account for small stiffness changes or geometric changes in the blades and even very large stiffness changes and geometric changes due to bends, dents, and blends. Typically, these analyses are done on single stage models and ignore the effects of the interaction of multistage and mistuning effects. Recently, a statistical analysis of small mistuning effects in multistage rotors has been conducted and shown the importance of multistage modeling. Additionally, a method to efficiently account for large mistuning in multistage rotors has recently been developed using Fourier Constrain Modes and Pristine Rogue Interface Modal Expansion (FCM-PRIME) methodology. The focus of this work is to better understand the combined effects of large and small mistuning on multistage rotors. This thesis will discuss how reduced order models of multistage rotors with both large and small mistuning can efficiently be created. It will also discuss the different effects of various types of large mistuning (e.g., dents, bends, and blends) on the multistage dynamics.

## Acknowledgements

I would like to thank the people who have supported this research project and the invaluable assistance I have received. Without their help, this work would not have been possible or even remotely successful. First I would like to thank Dr. D'Souza for taking a chance and giving me this opportunity to explore the interesting and always developing field of turbomachinery and computational research. Going into this project, I had no prior knowledge of this subject matter, but I now feel prepared and well integrated into the turbomachinery world with his guidance. I would also like to thank Eric Kurstak for his invaluable assistance in the technical trenches. Eric's insight and previous work with FCM-PRIME is fundamental to this project, and this would not be possible without his constant help. Lastly I would like to acknowledge the Gas Turbine Lab and all the innovative research and hard work conducted there.

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# Chapter 1: Introduction and Background

Turbine bladed disks are an integral component of jet engines and gas power generators, but are subject to severe vibration responses due to the strenuous operating conditions. Since the inception of the gas turbine engine, there has been great interest in identifying the causes and effects of these behaviors in order to mitigate potential structural damage to the engine that could potentially lead to catastrophic failure. For example, specific rotation speeds and frequencies could amplify the vibration of the turbine structure beyond safe tolerances and lead to fatigue or failure, and therefore should be avoided.

A major focus in this field is on the vibration effects due to damaged turbine stages. Any differences between blades is referred to as mistuning and comes from random variances in the structural properties of the blades or from damage caused by in service wear. A whole range of mistuning exists, from inconspicuous small mistuning (variances in structural properties) to overt damage (dents, bends, and missing portions). These deviations to a pristine turbine blade can cause localized energy in other blades on the disk or stages of the turbine. This leads to an amplified response that can cause high cycle fatigue,which can lead to engine failure[1].

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**Figure 1: Severely Mistuned Turbine Blades[2]**

Recently, advances in computational techniques have enabled large statistical analysis to investigate mistuning. Computational tools such as finite element analysis (FEA) have allowed for the modeling of the structure dynamics. These simulations calculate the modal and forced response of the turbine structure. However, primitive simulations of a multistage turbine with mistuning using traditional FEA techniques are quite time consuming when solving multiple parameters. This facilitates the need for reduced order models (ROMs) that decrease the time needed to conduct such an analysis. Often ROM methodsuse the cyclic symmetry of the turbine disk geometry in order to efficiently extract parameters from the FEA to construct the ROM. However a major challenge with mistuning is that it fundamentally destroys the symmetry of the bladed disk, requiring alternative methods to construct the ROM.

Several approaches have been developed to accurately and efficiently capture mistuning on the basis that the tuned system modes are a good basis for the mistuned system for small mistuning[3]. In this work, component mode mistuning[4] (CMM)is used to model the small mistuning, however CMM is unable to capture large mistuning or crack behaviorusing sector only computations and models. There has been recent research into accurately constructing models of bladed disks with cracks using linear approaches[5-8]. There has also been work focused on large mistuning using sector only calculations. These methods include the Pristine Rogue Interface Modal Expansion (PRIME) method[9], which can handle all manners of large mistuning including blends, dents, and large frequency deviations, and the mode accelerated X-Xr (MAX) method[10], which was developed for efficiently generating ROMs for systems with blends. PRIME is advantageous because it uses a single sector model of the bladed disk with compatibility conditions for pristine sectors (no large mistuning) and rogue sectors (with large mistuning). However the PRIME method alone is limited to single stage analysis. Previous work has also been done to construct multistage ROMsusing Craig-Bampton component mode synthesis method[11] (CB-CMS) with the interstage boundary kept active for each stage, while the rest of the system is reduced using fixed interface normal modes computed from sector level calculations[12-15]. More recent work with Fourier constraint modes (FCMs) has been done to more efficiently incorporate both small and large mistuning into multistage systems[16].

These computational methods have been used in past research to understand the effects of small mistuning on a two stage system. Using a statistical small mistuning analysis with the CMM method, it was concluded that multistage analyses are required to accurately capture the multistage vibration response, as a single stage analysis is insufficient. As the level of small mistuning is increased, there is sufficient amplification to the forced response in both stages from the tuned system. Lastly this work also highlights the need for efficient ROMs to predict the physical behavior of turbo machinery in a timely manner[13].

Recently, turbomachinery ROMs have been developed that integrate Fourier constraint modes and Pristine Rogue Interface Modal Expansion (FCM-PRIME) method to efficiently construct multistage ROMs with both small and large mistuning[16]. This method allows for any combination of large mistuning combined with small frequency mistuning for multistage systems. Ultimately, this new method allows for an efficientidentification of multistage frequencies and forced responses for systems with multiple stages. After the successful validation of this method, the FCM-PRIME method is now being used to conduct both small and large mistuning analysis on multistage systems for this research.

## 1.1 Focus of thesis

The focus of this thesis is to model the vibration response of several types of mistuning for a multistage turbine system. The mistuning consists of increasing severities of dented, bent, and blended (smoothed missing mass) blades, as well as the inclusion of random small mistuning of the blade frequencies. The mistuning is imposed on one of a two stage turbine blisk (one-piece bladed disk design), to capture the effects of mistuning for downstream turbine stages. The results will be used to compare the severity and characteristics of each mistuning case's frequency response.

Specifically, this thesis seeks to answer the following questions. What types of mistuning have a considerable amplified vibration response? How does the location and characteristics of large mistuning affect this response? What are the effects of small mistuning combined with large mistuning? How does mistuning in one stage affect other stages of the turbine?

## 1.2 Significance

The jet engine and turbomachinery industry is a specialized field with high tolerances for the structural and dynamic performance of gas turbines. This ensures a high level of safety and reliability for aerospace and power generation applications with little room for failure. Insight into the mistuning phenomenon is critical in the design process to reduce the occurrence of failure and fatigue prolonging the life cycle of the engine.

This research will further elaborate on the phenomenon of mistuning in turbomachinery, by contributing a new understanding of the effects of small and large mistuning on a multistage system. This new area of research builds upon past work with mistuning to provide a more complete perspective that accurately captures the entirety of a multistage system as opposed to only a single stage of the turbine. In addition this research will serve to validate the effectiveness of the newly created FCM-PRIME method as a research tool for turbomachinery dynamics.

## 1.3 Overview of Thesis

This thesis consists of 6chapters. The next chapter will provide an overview of the turbomachinery system and mistuning to be investigated. This will also be coupled closely with the methodology used to design the analysis of the mistuned models as referenced to in the following chapter. Chapter 3 will discuss the computational analysis used to conduct the vibration response, as well as the implementation of the FCM-PRIME methodology. Chapter 4 will organize the results of the multistage system with only large mistuning, while chapter 5 will focus on the combined effects of large and small mistuning. The results of the experiment will be used to form conclusions with a summary in chapter 6. In addition, a discussion of future application and the direction of research will follow.

## Chapter 2: Multistage Turbomachinery System and Mistuning

This chapter introduces the multistage turbomachinary system used to model the vibration response of a mistuned system. First, the multistage pristine system and its components are introduced as a finite element model, (figure 2). This pristine system is the basis for all added damage and mistuning, and is the foundation for any comparison. In addition, this chapter defines the various types of mistuning to be investigated and the quantitative metrics used to compare damage for each case.

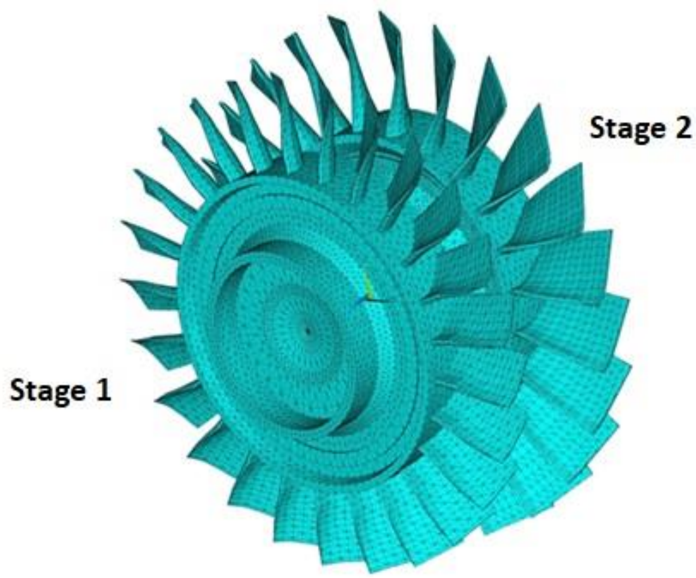


**Figure 2: Pristine Multistage Blisk**

This thesis considers both large and small mistuning. Large mistuning is applied as a single *rogue* (damaged) blade on the 2$^{nd}$ stage, while stage 1 remains pristine with no damage. Small mistuning is applied to all blades randomly in both stages. The methods used to create the mistuning damage are also discussed. The chapter concludes with a summary of all the mistuning parameters tested.

## 2.1 Tuned System

The pristine multistage system represents the ideal turbine as it was designed for service. This pristine or *tuned* system was specifically created to minimize the deflection of its blades due to the oscillatory forces experienced at the turbine's operating condition. In this perfect state, every blade on each stage is identical and free from damage. The material properties, represented as the mass and stiffness of the finite element model, are consistent throughout the system.

The multistage system consists of two unique bladed disks connected at the shaft's interface. Stage 1 has 25 blades while stage 2 is downstream of stage 1 and has 23 slightly larger blades. For the pristine system, each *individual* stage has the property of cyclic symmetry about the center of the shaft, but the coupled two-stage system does not have this property. Every blade also has a corresponding section of disk that comprises the sector level model, (figure 3). For example, the tuned $2^{nd}$ stage consists of 23 identical bladed disk sectors rotated around the center. This is a very important quality as it allows for cyclic analysis to generate the ROMs, assuming all sector models are pristine. This is further elaborated on in the discussion of the FCM-PRIME method in chapter 3.

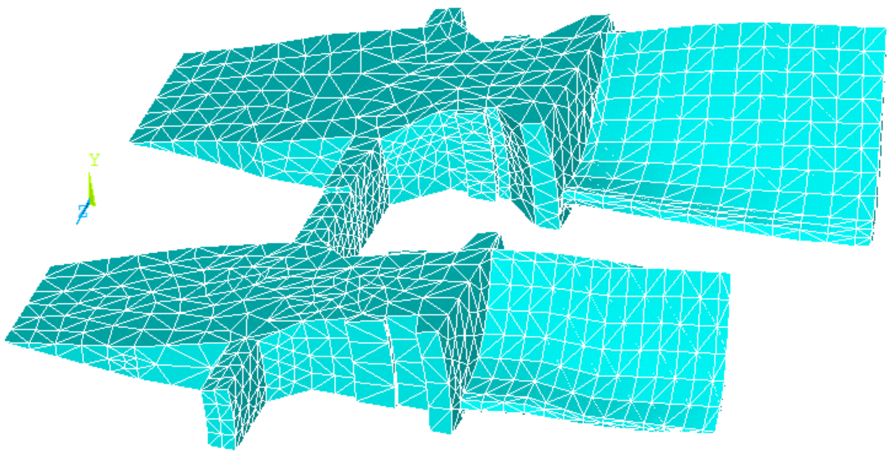


**Figure 3: Pristine Sector Level Models**

The behavior of the multistage pristine system with no mistuning is analyzed for reference (figure 4).A forced response over a frequency range of 1.5-4.5 kHz was conducted with a point force applied to the tip of every blade in the flow direction on both stages. This frequency range was selected because it encompasses the most active response the turbine will experience in operation. The maximum deflection amplitude of any blade was recorded for each frequency with an engine order of 1 for simplicity.

The pristine stage 1 turbine has a peak deflection of .00729 mm at a frequency of 3.23 kHz with a second smaller peak occurring at 2.91 kHz. The pristine stage 2 has a single larger peak deflection of .0083 mm at 2.19 kHz. These pristine values are used for comparison with the mistuned systems to determine the max amplification in the mistuned responses, as this is the most deflection any blade on the turbine stage will experience.

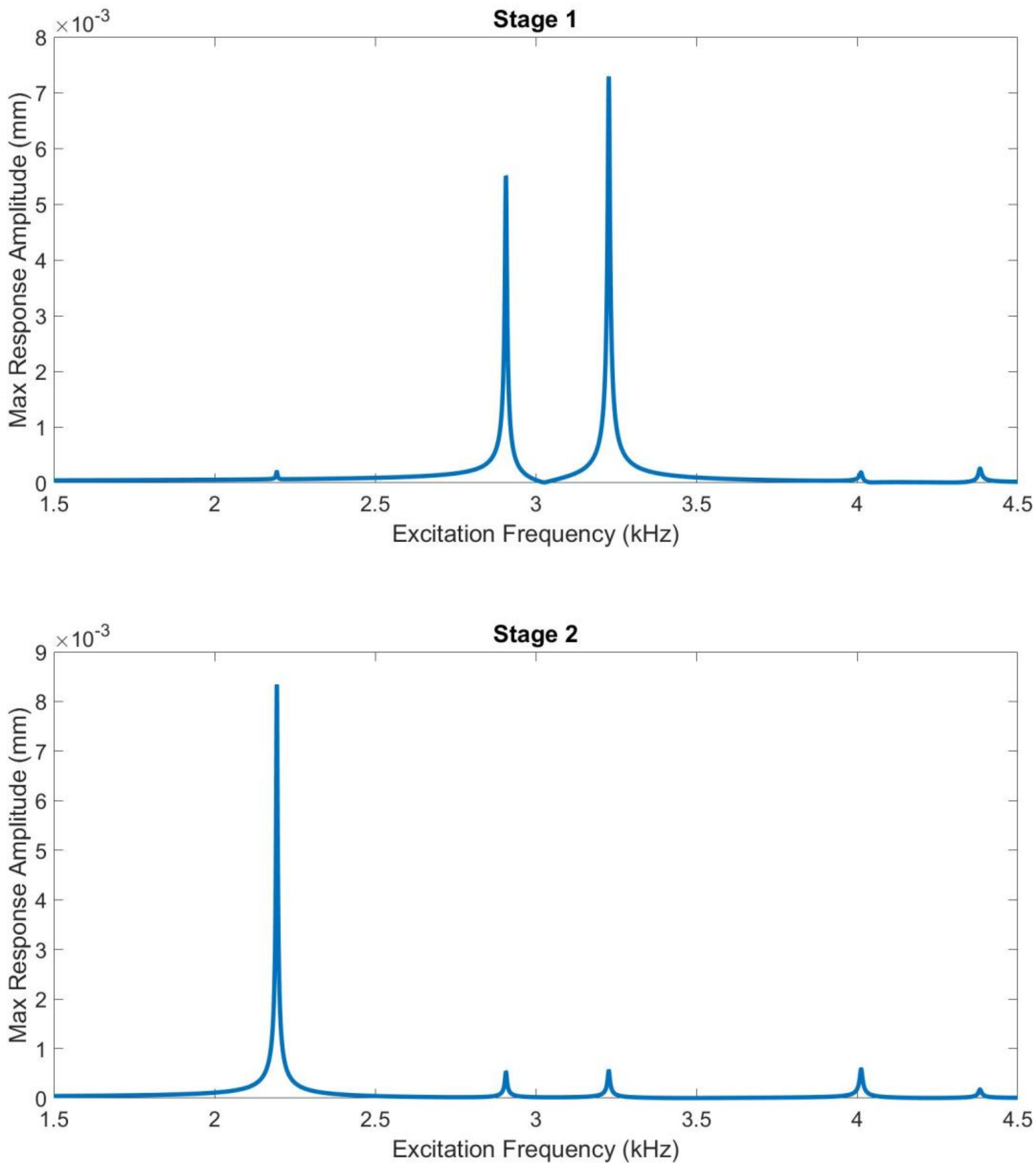


**Figure 4: Pristine System Forced Response**

## 2.2 Large mistuning

The pristine system is the design target, but realistically it is rarely achieved in manufacturing and cannot be maintained throughout the engine's lifecycle. In the next two sections, large and small mistuning is defined, as well as the various unique types of damage and their characteristics. In the context of turbomachinery, mistuning by definition eliminates the cyclic symmetry of a turbine stage. The blade to blade difference can come in many forms, but fundamentally mistuning is any deviation from the pristine blade design. The large mistuning presented in this research is limited to any deformation or missing portion of the pristine blade's geometry. The small mistuning is used to model the random inherent differences in the blades when they are manufactured.

Large mistuning is usually visibly identifiable damage and occurs in two distinct cases. The first case is *deformation* damage. Two types of deformation damage are examined, *dents* and *bends* to the blade. Dents are most commonly the result of the impact of small foreign object debris in the engine (figure 5). Dents are characterized by a localized deformation in the blade, while the overall geometry of the blade remains unchanged. This analysis will focus on three dent locations in the middle portion of the blade: near the root, the center, and near the tip. Each dent has an impact diameter 1/3 the span of the blade, with the greatest displacement at the center.

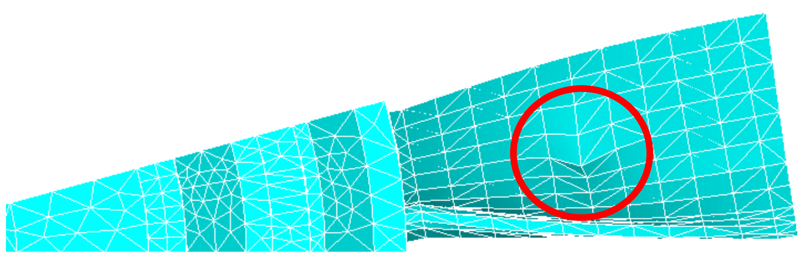

**Figure 5: Stage 2 Dented Blade (Center 5% LMP)**

In the case of a bent deformation, the entirety of the blade is deformed due to a bending moment applied at the tip of the blade. This damage is most likely the result of larger debris impacting near the tip of the blade. Three directions of bends are examined: the upward and downward bends in the tangential direction of rotation, and a backward bend in the direction of the flow.

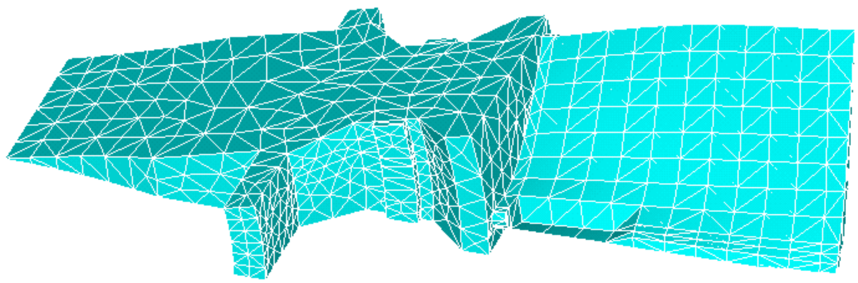

**Figure 6: Stage 2 Blended Blade (Root 5% LMP)**

The second case of large mistuning is missing mass. Missing mass models are characterized by damage to portions of the blade that have removed material such as nicks or pits due to the impact of debris. Moreover this damage is often repaired with *blends* that intentionally shave off material to smooth the damaged area and reduce points of high stress (figure 6). Since turbine blisks are continuous between the blade and disk, the damaged blade cannot be removed. Therefore it is important to understand how these blended modifications affect the dynamics of the blade. The blends examined are applied to the leading edge of the blade near the root, center and around the tip.

Another common form of damage considered as large mistuning is that of a crack or splits to the blade. This introduces non-linear dynamics that are outside the scope of this research, and will not be addressed, but are also important.

## 2.3 Large Mistuning Percentage

The different types of mistuning outlined above are described in qualitative terms for characterizing the type of damage. However, in order to analyze vibration behavior corresponding to the severity of damage in each case, it is important to define the Large Mistuning Percentage (LMP) in quantitative terms. LMP is the relative percentage of mistuning severity andis defined uniquely for deformation and missing mass due to the different nature of the damage.

The LMP for deformation mistuning such as bends and dents is defined as the ratio of the max displacement (d) from the pristine blade normalized by the length of the blade (L) measure at the leading edge, (figure 7 and equation. 1).The LMP for missing mass large mistuning is defined as the ratio of removed mass to the total mass of the blade. For the finite element model, the removed mass is measured by the number of elements (E) removed in the blade of the sector model as shown in eq.2. Three models for LMP values of 1, 3, and 5% are created for each mistuning case to analyze how the response changes with more severe damage.

$$LMP_{deform} = \frac{d}{L} \times 100 \qquad \textit{Equation 1}$$

$$LMP_{mass} = \frac{E_{missing}}{E_{blade}} \times 100 \qquad \textit{Equation 2}$$

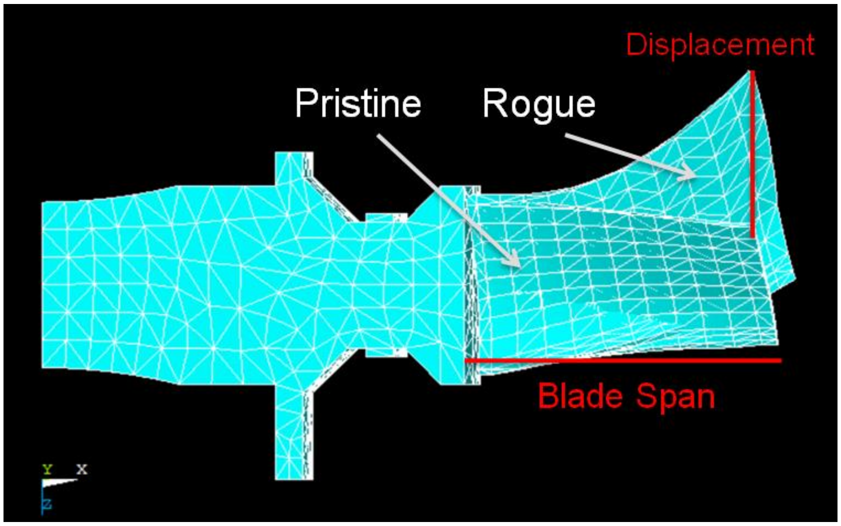


**Figure 7: Deformation Large Mistuning Percentage for Exaggerated Bend**

An example of how LMP is used is demonstrated by the forced response shown in figure 8. This figure shows the 2nd stage response when a single blade is damaged by an upward bend. The increased damage, measured as LMP, amplifies the response and creates new smaller peaks at nearby frequencies. This research is interested in how, and to what extent the shape of the forced response changes with the LMP. In this particular case small deviations at LMP of 1% amplify into larger more profound peaks as LMP increases. The ratio of the deflection amplitude at a particular frequency for the mistuned system compared to the pristine system is referred to the amplification factor. In particular, this research is interested in the peak response for each LMP and the maximum amplification factor across all frequencies. This forced response and amplification analysis is carried out for all mistuning scenarios for both stages in order to understand the multistage effects of mistuning.

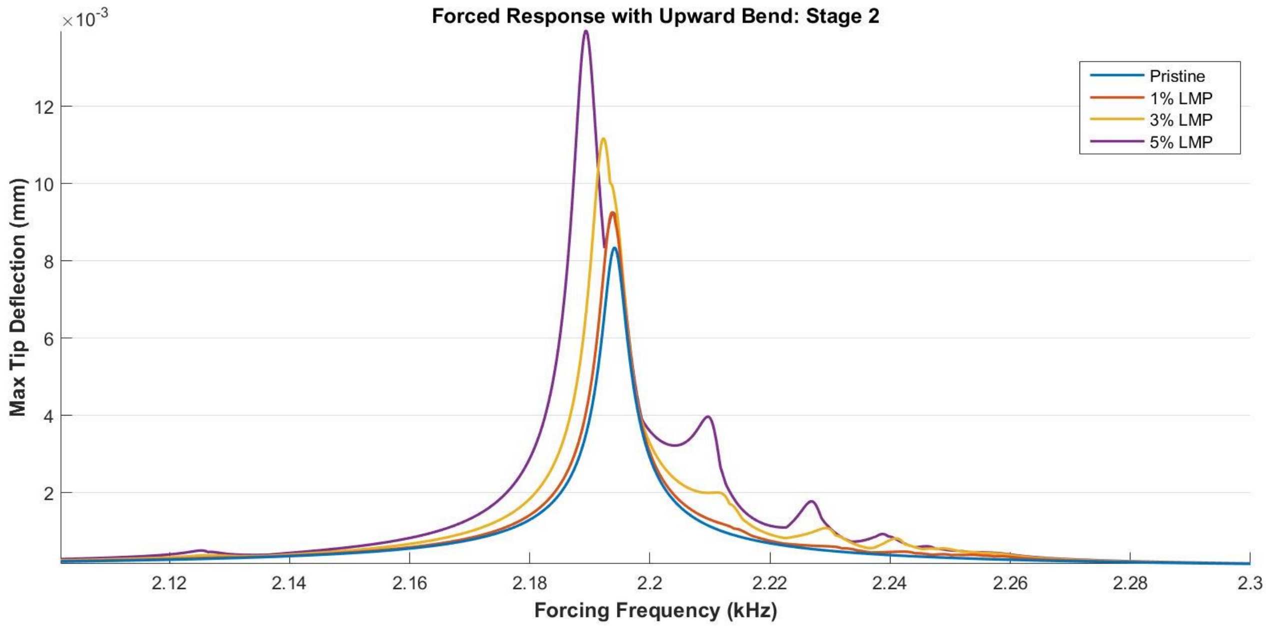


**Figure 8: Stage 2 Forced Responses for Upward Bent Blades (1.8-2.6 kHz)**

## 2.4 Small Mistuning

Small mistuning is also studied in this research and can be the result of in-service wear or manufacturing tolerances. Due to the random nature of the differences in each blade, statistical methods are needed to analyze small mistuning. A mistuning pattern is applied to pristine blades for stage 1 and stage 2 sector models. This mistuning pattern allows for the adjustment of the material properties (Poisson's ratio, Young's modulus, density) at selected nodes and elements throughout the blade. These properties are varied statistically using a Gaussian distribution with a standard deviation of 4% around the pristine material property values. The greater the deviation, the more severe the small mistuning effects are.  All small mistuning patterns are used at a moderate 4% small mistuning level.

This ultimately yields random changes in the mass and stiffness values of the sector models in the blade portion and therefore the natural frequencies and forced response. This is the process used to create a single randomly generated small mistuned forced response. Integrating this small mistuning ROM into FCM-PRIME increases the speed of calculating a single forced response by an order of magnitude.This allows the process to be easily repeated 100 times to determine the maximum likely deflection for this particular level of small mistuning.

In addition, small mistuning can be combined with large mistuning utilizing the full potential of FCM-PRIME. 100 random small mistuning patterns were generated for each type of large mistuning with 5% LMP to simulate the worst case. In this scenario, again large mistuning is only applied to a single blade in the 2nd stage, while small mistuning occurs randomly in all blades of both stages. This research is interested in understanding how large mistuning in one stage, combined with small mistuning in all blades, can affect other stages of the system.

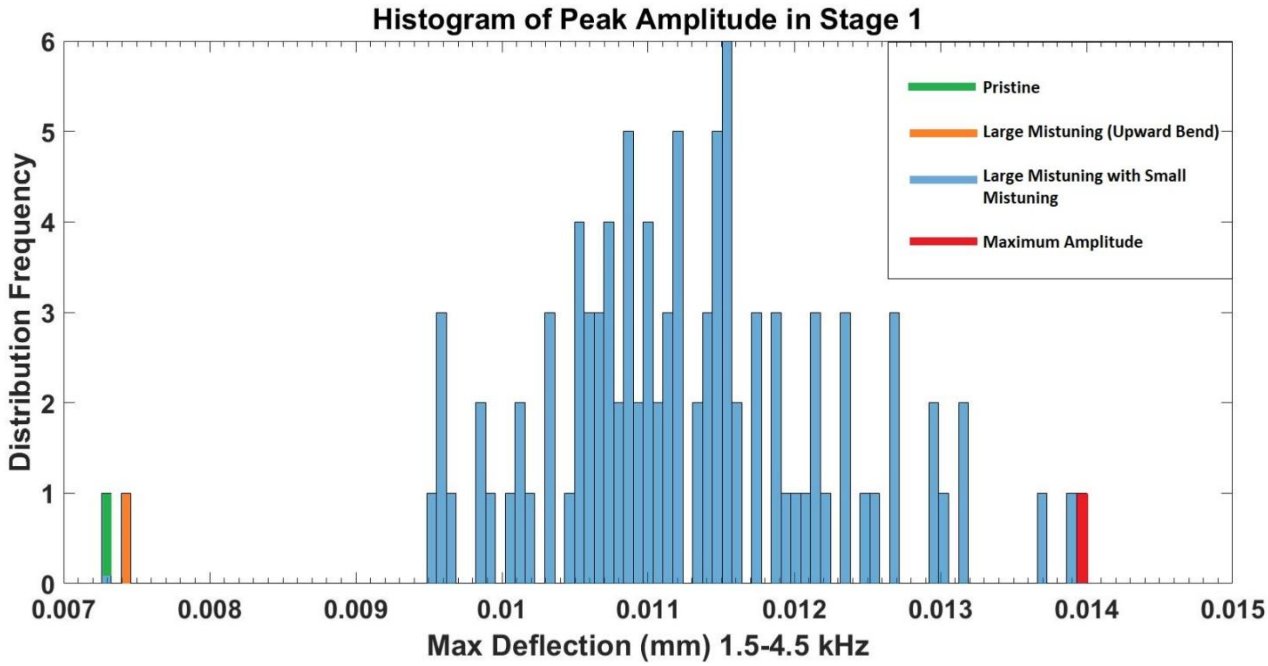


**Figure 9: Stage 1 Small Mistuning Distribution of Peak Deflections (Upward Bend 5% LMP)**

The stage 1 results of 100 small mistuning patterns, combined with large mistuning (upward bend 5% LMP) in stage 2 are shown in figure 9. The peak deflection of each analysis is plotted showing the distribution of the results for all 100 cases. Even with the relatively small sample size of 100 patterns, a distribution appears that gives insight into the average and maximum amplitude of all 100 tests. In addition the maximum deflection amplitude for the pristine case, and large mistuning only case are shown for comparison. This kind of analysis is what is used to understand and compare the amplification factor of each type of mistuning that will be discussed in the results of chapter 5.

## 2.5 Summary of Mistuning

Each type of large mistuning considered is analyzed independently and in combination with small mistuning. Not only are the types of mistuning of interest, but so is the severity and location of mistuning. Figure 10 shows the location of all types of large mistuning, and Table 1 summarizes all mistuning parameters to be tested and compared.

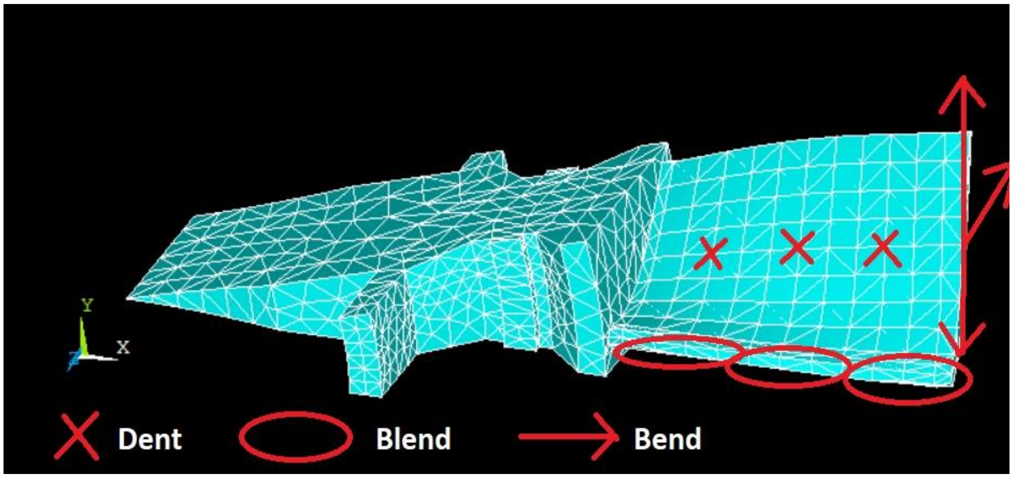


**Figure 10: Large Mistuning Location**

| Bent Blades | | | |
|---|---|---|---|
| Direction | Upward at tip | Downward at tip | Backward at tip |
| Large Mistuning Percentage | 1% | 1% | 1% |
| | 3% | 3% | 3% |
| | 5% | 5% | 5% |
| | 5% w/ small mistuning | 5% w/ small mistuning | 5% w/ small mistuning |

| Dented Blades | | | |
|---|---|---|---|
| Location | Center root | Center | Center tip |
| Large Mistuning Percentage | 1% | 1% | 1% |
| | 3% | 3% | 3% |
| | 5% | 5% | 5% |
| | 5% w/ small mistuning | 5% w/ small mistuning | 5% w/ small mistuning |

| Blended Blades | | | |
|---|---|---|---|
| Location | Leading edge root | Leading edge center | Leading edge tip |
| Large Mistuning Percentage | 1% | 1% | 1% |
| | 3% | 3% | 3% |
| | 5% | 5% | 5% |
| | 5% w/ small mistuning | 5% w/ small mistuning | 5% w/ small mistuning |

**Table 1: All Mistuning Parameters Tested**

## Chapter 3: FCM-PRIME Methodology

This chapter explains the methodology of FCM-PRIME and its implementation of mistuning to find the forced response of each system in a computationally efficient manner. It will also go into the mechanics of FCM-PRIME to discuss necessary information needed to perform this reduction, and also highlight its advantages. First, we begin by again emphasizing the problems mistuned multistage systems create that drive the need for innovative methods like FCM-PRIME. Most obviously the size of the system has more than doubled with the addition of the second stage greatly increasing the number of computations. When both stages are combined, the symmetrical property that each individual pristine stage has is no longer true. Even worse, when adding mistuning, the cyclic symmetry of these individual stages is also destroyed.

To demonstrate the effectiveness of cyclic symmetry in computational modeling, this particular two-stage system has a total of 162,549 DOFs in the full model. However using cyclic expansion, the same system can be represented with just two sectors models of 3063 and 3738 DOFs for stage 1 and stage 2, respectively. This reduction is of great advantage and therefore the pristine system is analyzed using sector level models. Methods like CB-CMS have already allowed for this multistage modeling of pristine systems. However, in order to incorporate small and large mistuning into a multistage system, FCM-PRIME must be used

## 3.1 FCM-PRIME

This section gives background to the FCM-PRIME methodology and its implementation. As mentioned before, FCM-PRIME is a reduced order model the can efficiently conduct a modal and forced response analysis for a multistage turbine system with mistuning. There are two major components of FCM-PRIME that are combined to reduce the system. The Fourier Constraint Modes (FCM) that couple the dynamics between multiple stages of the turbine, and the Pristine Rogue Interface Modal Expansion (PRIME) that uses the cyclic symmetry analysis with single stage sector models to incorporate the effects of large mistuning into the stage model. Together, the individual ROMs are integrated to greatly reduce the number of degrees of freedom (DOF) that are needed to solve the multistage mistuned system.

### 3.1.1 Fourier Constraint Modes (FCM)

Traditionally, multistage modeling is carried out by computing fixed interface constraint modes on the interface DOFs between two stages while computing the interior DOF mode shapes. As opposed to these single perturbations on a DOF-by-DOF basis, Fourier constraint modes are calculated with groups of active DOF perturbations on the inter-stage boundary. This requires the use of a Fourier displacement set**,** which is unique to each stage. This provides a single Fourier basis that can project the boundary node motion of each stage. The Fourier displacement set **F** is used to create the mode shape of the boundary between stages which become the Fourier constraint modes[16].

Each stage has a different number of DOFs at the interstage boundary due the difference in sectors of each stage. However the DOFs in a specific direction of each stage can be related if the nodes lie on concentric rings, through shared harmonics. This system is comprised of 3 concentric rings on the interstage boundary. The Fourier displacements for the DOFs on given ring and directions are assembling in the Fourier displacement matrix

$$\boldsymbol{F} = [\boldsymbol{x}_{\mathbf{1,1}} \boldsymbol{x}_{\mathbf{1,2}} \boldsymbol{x}_{\mathbf{1,3}} \dots \boldsymbol{x}_{\mathbf{3,1}} \boldsymbol{x}_{\mathbf{3,2}} \boldsymbol{x}_{\mathbf{3,3}}] \qquad \textit{Equation 3}$$

Where **x** is a column vector of the displacement for all the DOFs on the interstage boundary for a stage. The first subscript corresponds to the ring and the second subscript to the DOF direction. This set of vectors is then used to compute the constraint modes $\boldsymbol{\Psi}$ the system. Now the stage can be transformed into reduced coordinates $\boldsymbol{p}$ with the additional use of the normal modes $\boldsymbol{\varphi}$ of the stage as shown below. The subscript **b** denotes the DOFs on the boundary and **s** denotes the DOFs of the rest of the stage ***i.***

$$\begin{bmatrix} \boldsymbol{u}_b^i \\ \boldsymbol{u}_s^i \end{bmatrix} = \begin{bmatrix} \boldsymbol{F}^i & 0 \\ \boldsymbol{\Psi}^i & \boldsymbol{\varphi}^i \end{bmatrix} \begin{bmatrix} \boldsymbol{p}_{\boldsymbol{\Psi}}^i \\ \boldsymbol{p}_{\boldsymbol{\varphi}}^i \end{bmatrix} \qquad \textit{Equation 4}$$

Ultimately, the benefit of the FCM method comes from reduction of inter-stage constraint modes that need to be calculated. This technique also allows for the disregard of meshing differences between stages, provided that the interface nodes are in concentric rings. In addition FCMs can be calculated using sector level models for each model used in the system. Consequently, FCM is versatile and can be combined with other ROMs that use cyclic sectors to reduce the multistage system before the FCM is applied.

### 3.1.2 Pristine Rogue Interface Modal Expansion (PRIME)

The Pristine Rogue Interface Modal Expansion method analyzes small and large mistuning on a sector level for a single stage system. PRIME exploits the cyclic symmetry of each stage, using the pristine sector models and any combination of mistuning or rogue sectors.

The PRIME reduction requires the normal mode shapes $\boldsymbol{\varphi}^i$ and geometric data (mass and stiffness matrix, **M** and **K**) from each sector model used in the stage. Since the rogue models are derived from the pristine model, all interface DOF arrangements are consistent and therefore can be applied cyclically. The final PRIME transformation to reduce a single stage with mistuning is shown below.

$$\boldsymbol{T}^{\boldsymbol{PRI}} = \begin{vmatrix} \boldsymbol{\varphi}^{\boldsymbol{P}}\boldsymbol{0} \\ \boldsymbol{\varphi}^{\boldsymbol{R}}\boldsymbol{\varphi}^{\boldsymbol{R}}\boldsymbol{V}^{\boldsymbol{n}}\boldsymbol{V}^{\boldsymbol{r}} \\ \boldsymbol{\varphi}^{\boldsymbol{I}}\boldsymbol{0} \end{vmatrix} \qquad \textit{Equation 5}$$

Here the superscript ***P, R*** and ***I*** refer to the normal modes of the Pristine, Rogue, and Interface DOFs, respectfully. In order to make this matrix well-conditioned and kinematically admissible, the range space $\boldsymbol{V}^{\boldsymbol{n}}$ and null space $\boldsymbol{V}^{\boldsymbol{r}}$parameters must also be specified. In cases of large mistuning only, a satisfactory range space of 1e-16 and null space of 1e-7 were used for each stage. Cases of mixed mistuning used a range space of 1e-13 and null space of 1e-10 for each stage.

After the PRIME reduction is carried out for each stage, it can be combined with FCM to reduce the multistage mistuned system. This final step requires the partitioning of the pristine, rogue, and interface constraint mode shown below in Equation 6. The full FCM-PRIME reduction is also shown below.

$$\boldsymbol{\Psi}^{PRI} = \begin{bmatrix} \boldsymbol{\Psi}^{P} \\ \boldsymbol{\Psi}^{R} \\ \boldsymbol{\Psi}^{I} \end{bmatrix} \qquad \textit{Equation 6}$$

$$\begin{bmatrix} \boldsymbol{F} & \boldsymbol{0} \\ \boldsymbol{\Psi}^{\boldsymbol{PRI}} & \boldsymbol{T}^{\boldsymbol{PRI}} \end{bmatrix} \qquad \textit{FCM-PRIME reduction}$$

The full FCM-PRIME integrates the PRIME reduction $T^{PRI}$,into the multistage stage Fourier constraint ROM as shown above. This transformation reduces the size of **M** and **K** by 99% allowing for quick calculation of eigenvalues and eigenvectors. Ultimately, FCM-PRIME is a tool that allows for efficient modal and forced response calculations using these results to model the multistage system with large mistuning.

## 3.2 Methodology and Requirements

The general methodology of this work is conducted as follows. First the mistuning cases of interest must be defined and modeled. Next, preliminary data unique to each model must be extracted from ANSYS, using traditional FEA techniques. This data is then inputted into the FCM-PRIME algorithm in order to perform a modal and forced response analysis. The results from each mistuning case and level are then recorded, analyzed, and compared to draw conclusions from.

In order to create the FCM-PRIME ROM, five key data inputs are required from each sector model included in the analysis. They are as follows: the list of interface nodes between stages, mass and stiffness matrices of each sector model used (2 pristine for each stage and the rogue), mode shapesof each sector model, and the Fourier constraint modes used to couple both stages. If small mistuning is also considered, the mistuning pattern and cantilever blade modes are also needed for every sector model used.

### 3.3 FCM-PRIME Validation

The computational savings of FCM-PRIME are evidently clear, but the accuracy of the ROM is also just as important. Every large mistuning only case with a LMP of 3%, and one small mistuning case for each type of damage was compared to the full FEA results using ANSYS. Figure 11 is an example of the modal analysis error of FCM-PRIME for the case of an upward bend large mistuning. The results validate the ROM with a maximum error of 0.012% discrepancy between the natural frequencies calculated. This is sufficient to confirm the accuracy of this solution. Similar agreement was also found for other large mistuning cases.

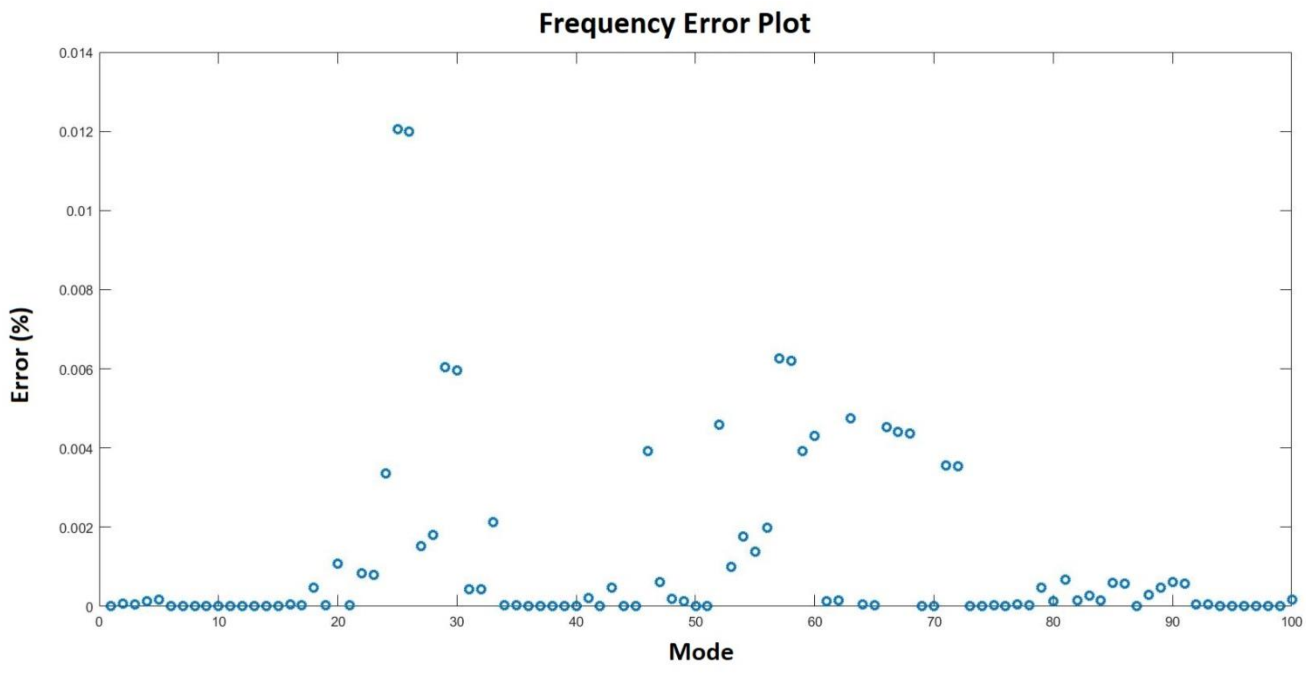


**Figure 11: FCM-PRIME Modal Analysis Error (Upward Bend 5%LMP)**

# Chapter 4: Multistage Large Mistuning Results

This chapter explores the results of large mistuning specified in section 3.3 and discusses the amplification of the vibration response on the two-stage system. The first section analyzes the multistage system with only large mistuning on stage 2. The results primarily draw conclusions by comparing the peak responses and maximum amplification across the frequency range targeted. Additional analysis is sometimes conducted by looking into the forced response of particular large mistuning cases that show interesting results. The primary focus of this research is for looking into the multistage effects of mistuning (stage 1 results), but will also examine results of both stages.

All vibration behavior is initially tested with a forced response across excitation frequencies from 1.5 kHz to 4.5 kHz using the FCM-PRIME method. A point force with an arbitrary magnitude of 100 N is applied to each blade tip at an engine order of 1. The maximum displacement of all nodes is calculated at each frequency step, forming the full frequency sweep response. The forced responses are carried out for each set of mistuning parameters for each stage. These forced responses are then used to compare the peak response and amplification in the forced response with mistuning.

In the following sections, the forced response similar to figure 8 in section 2.3 was conducted, now including a single rogue blade to the 2$^{nd}$ stage. Only one case of large mistuning is tested at a time on a single blade in stage 2. These forced responses are carried out for each large mistuning case at LMP's of 1, 3, and 5%. The peak displacement across the frequency range is recorded and plotted as a function of LMPfor both stages. Each type of large mistuning case (Bends, Dents, and Blends) also compares their unique characteristics such as location or bend directions.

### 4.1 Bent Blades

Bent blades are the first type of large mistuning to be examined with an upward, downward, and backward bend each with LMP of 1%, 3%, and 5%. The peak deflection for each large mistuning parameter is recorded for both stages, (Figure 13). The general trend shows that for all bending directions, the peak response increases as LMP increases. Also as expected, stage 2 (where the rogue blade is applied) has a much greater peak amplification factor of 1.67. In contrast stage 1's peak amplitude has a peak amplification factor of 1.018. In both stages it is observed that an upward bend is the most unfavorable, particularly for stage 2, with a max response 19% greater than the backward bend, and 39% greater than the downward bend at a max LMP of 5%.

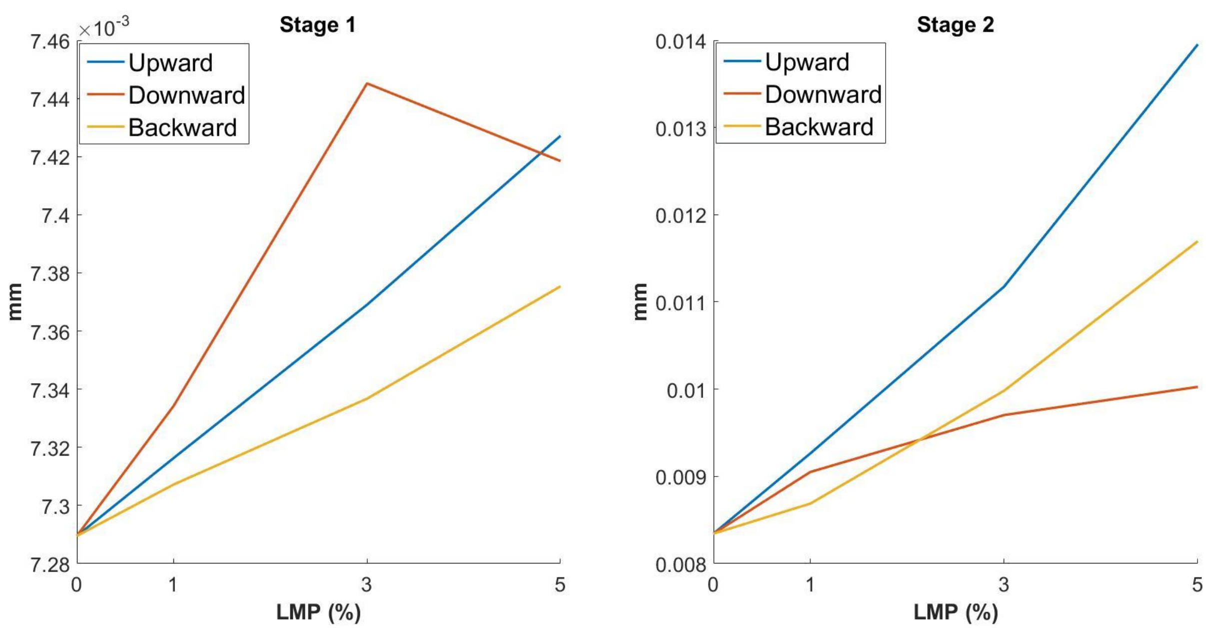


**Figure 13: Peak Response for Bent Blades (1.5-4.5 kHz)**

Another point of interest with large mistuning is the sensitivity of the peak response as the severity of the bend as LMP increases. In stage 1, the upward and backward bends have a near constant increasing slope, while the downward bend actually decreased from 3-5% LMP. Stage 2, has a similar trend with the downward bend's peak response tapering off as LMP increases. The upward and backward bends have an increasing slope, and are therefore more sensitive to the severity of the bend.

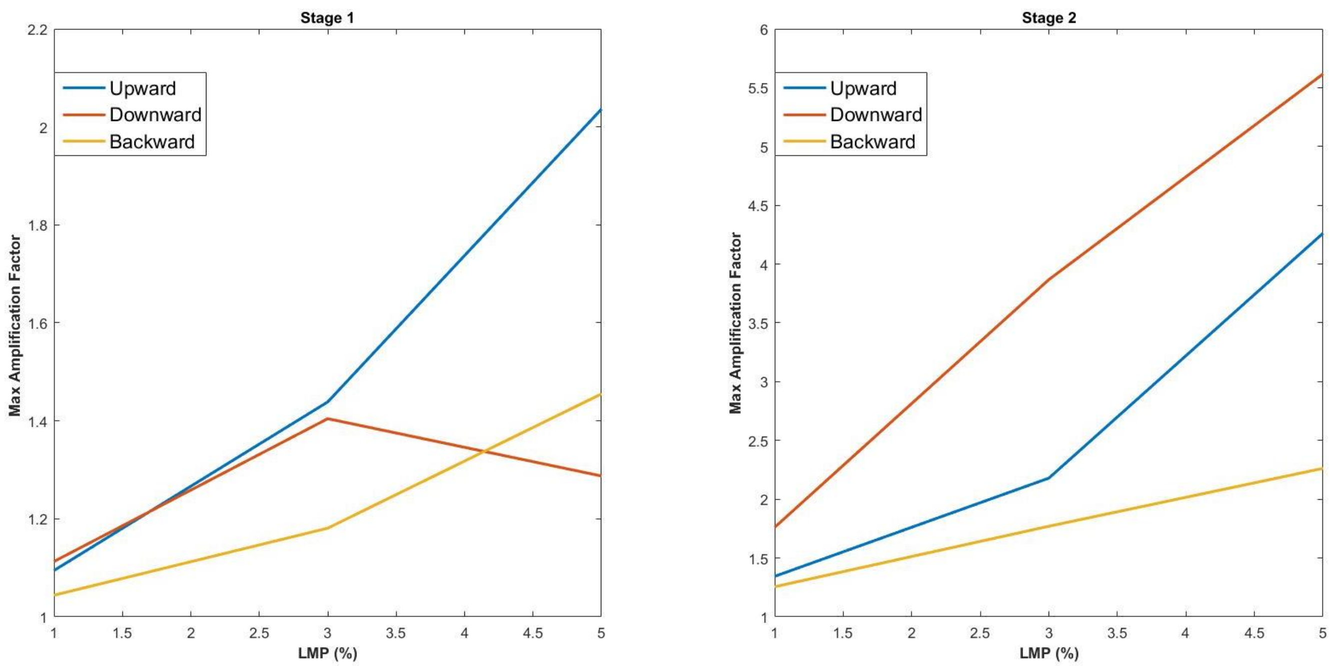


**Figure 14: Max Amplification Factor for Bent Blades (1.5-4.5 kHz)**

The maximum amplification factor across all frequencies is recorded in figure 14. This gives helpful insight into the behavior because the maximum amplification does not necessarily occur at the peak frequency. This can be seen in the downward blade which has the lowest peak response but has the highest amplification factor at LMP of 5%. However this large amplification is noteworthy at a frequency of 2.26 kHz and amplitude of .00129 mm, as a new peak forms. This could potentially be of concern if the engine operates at this frequency, leading to high cycle fatigue problems. This amplitude is relatively small compared to the peak amplitude of the pristine of .00831 mm at 2.19 kHz, which is 5 times greater. Although the maximum amplification factor does not directly correlate to the peak response, it does give the relative amplification across the entire frequency range. This point is further exemplified in blended blades, as its effects are smaller for bends and is insignificant in the second stage.

### 4.2 Dented Blades

The dented blades show similar trends to the bent blades but with milder amplification (figure 15). A dent at the root yields the greatest peak response in stage 2 at the highest LMP, with a 16% increase. Stage 1's worst case peak amplification is even smaller at only 0.6%. Once again the majority of the amplification due to large mistuning takes place in stage 2 where the dent is applied. Observations of the dent's location show that the closer the dent is to the disk, the more sever the response is for that stage. This trend is also present in stage 1, but greatly diminished in strength.

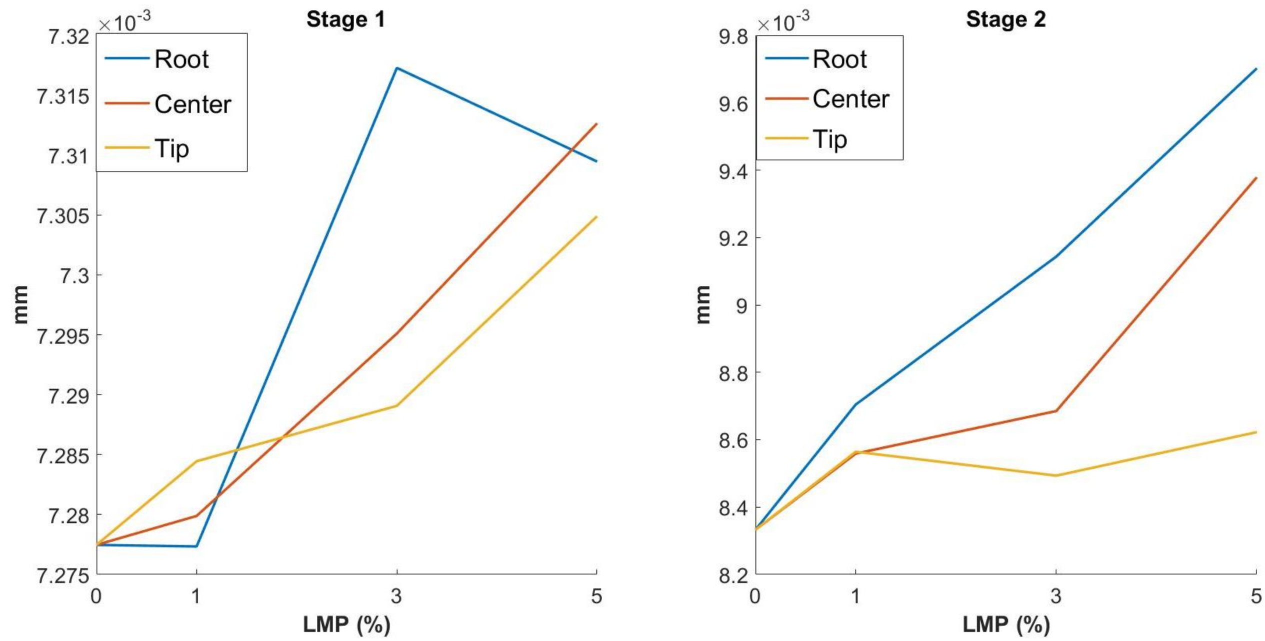


**Figure 15: Peak Response for Dented Blades (1.5-4.5 kHz)**

The maximum amplification factor across all frequencies also gives interesting insight, (figure 16). Despite the general decrease in peak response compared to bent blades, the max amplification factor is much greater for both stages. Again, this does not directly correlate to any significant peak response, but does show that certain frequencies away from the peak are being excited. For the worst case dent at the root (5% LMP), also has the largest amplification factor of 7.04 but just like with the bent blade, the response in negligible compared to the peak response.

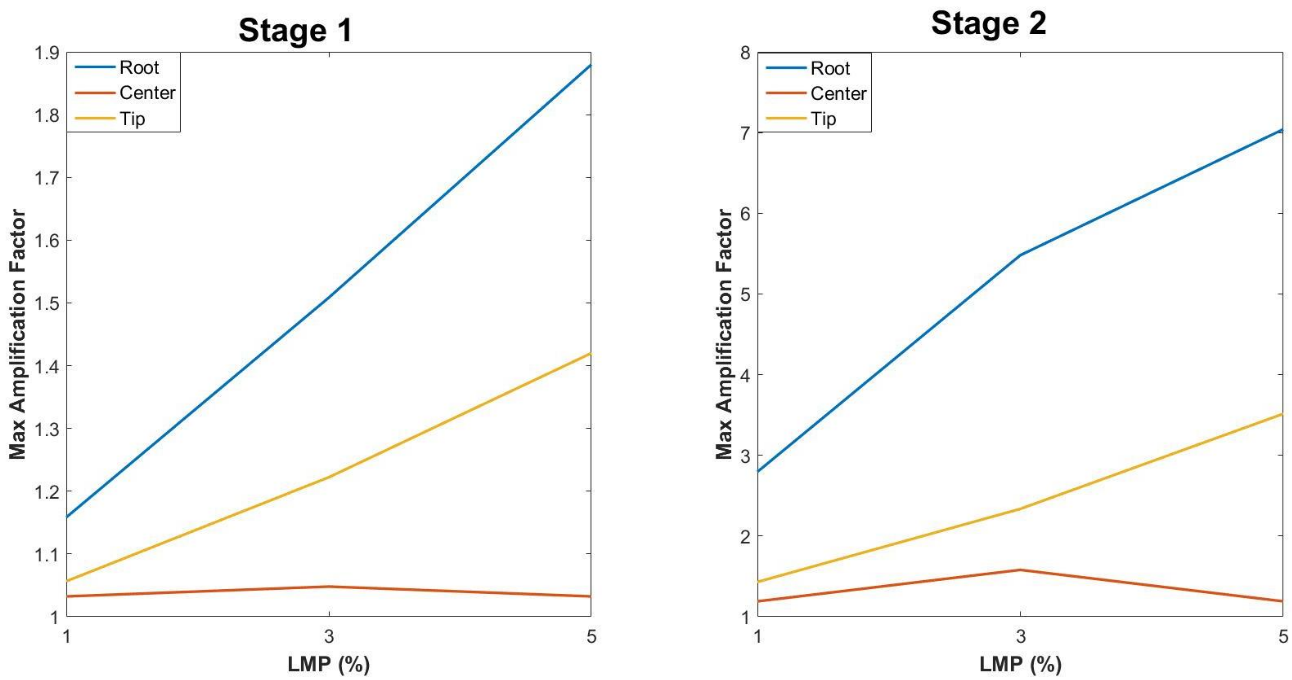


**Figure 16: Max Amplification Factor for Dented Blades (1.5-4.5 kHz)**

## 4.3 Blended Blades

The results from the blended blades have a much larger peak response compared to deformation mistuning. This trend is found in both stages, but as in the case of deformation, the response is much greater in the second stage, (figure 17). In stage 2, blends on the leading edge at the root and center of the blade have the greatest peak amplification of 73.6% and 64.3% respectively, compared to the pristine.  These locations also have a greater response as a function of LMP than the blend near the tip, which tapers off as LMP increases. In fact the maximum amplification factor across all frequencies decreases as LMP increases for the tip location, (figure 18). This is contrary to the root and center which skyrocket to amplification factors of 130.6 and 80.85 respectively.

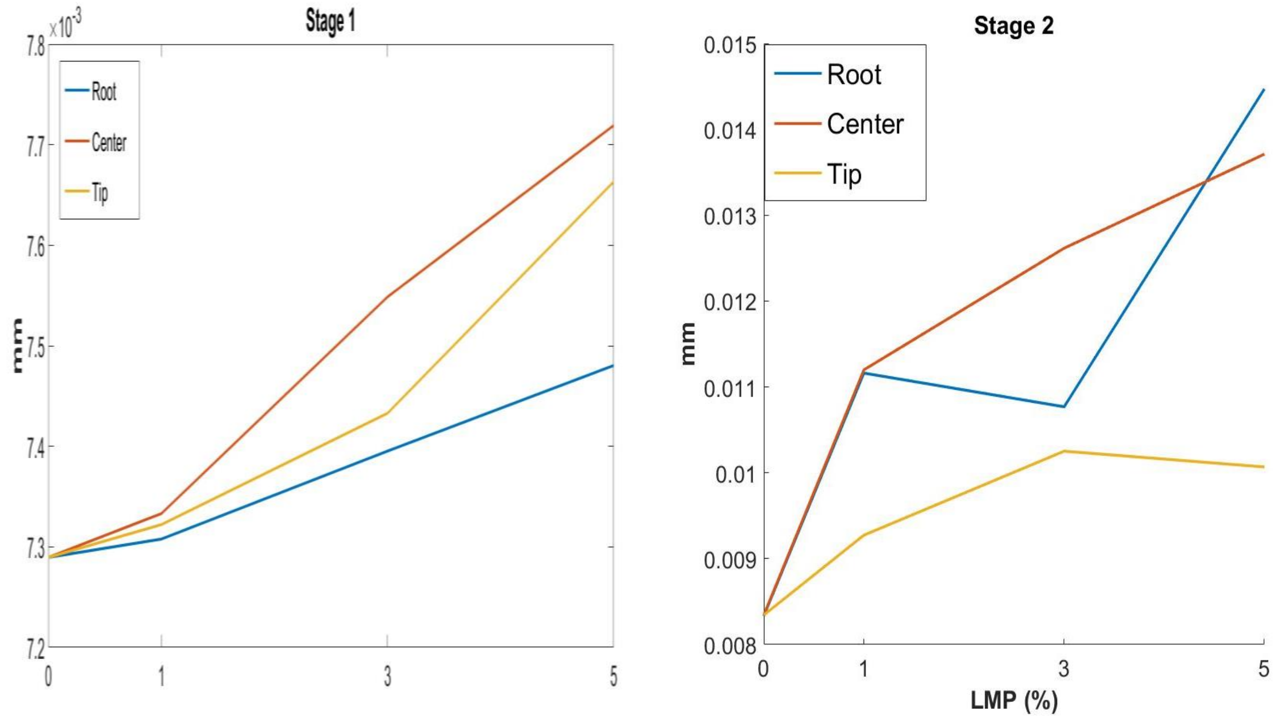


**Figure 17: Peak Response for Blended Blades (1.5-4.5 kHz)**

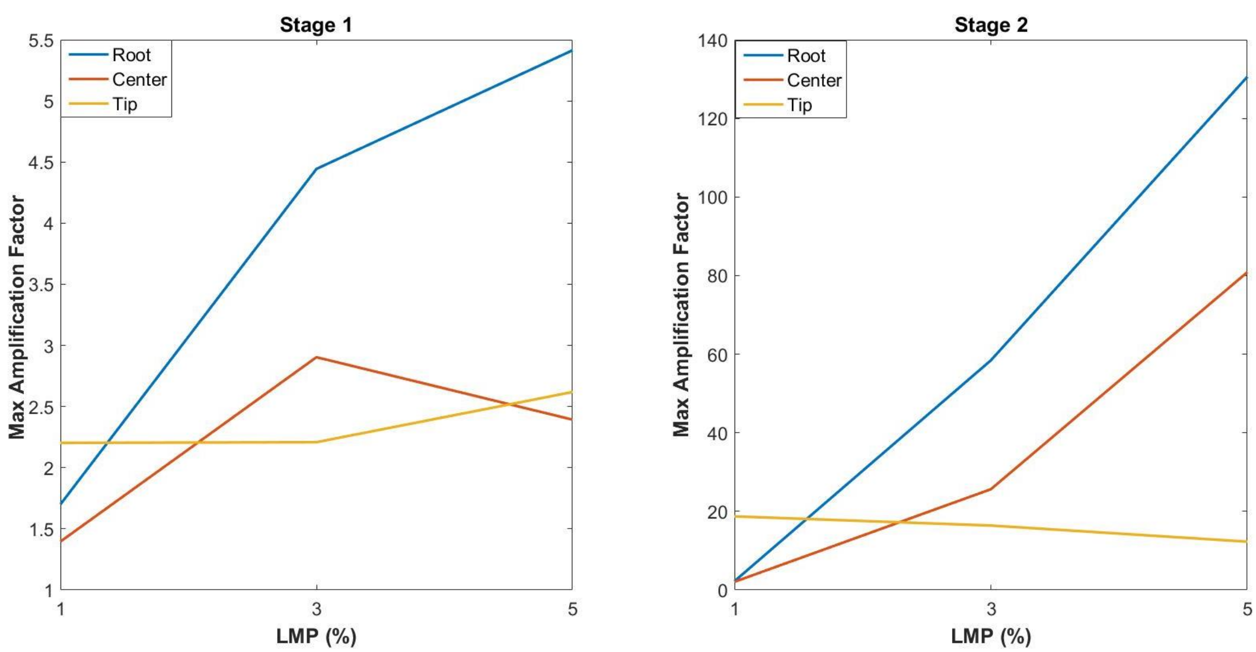


**Figure 18: Max Amplification Factor for Blended Blades (1.5-4.5 kHz)**

Similarly to bent blades, the max amplification occurs at frequencies near the peak. However in the case of blends the results are more significant. Inspection of figure 19shows the max response across all frequencies in the 2nd stage with a blend at the root. As LMP increases, the peak at 2.19 kHz grows and eventually splits into two major peaks, each larger than the pristine response. Figure 20 shows how the amplification factor shifts and grows to lower frequencies with the new peak forming at 2 kHz. This is a significant result in the second stage, because a wider range of frequencies should be avoided due to the significant amplification. The same splitting of peaks and large amplification of lower frequencies is also observed in the case of the blend to the middle of the leading edge.

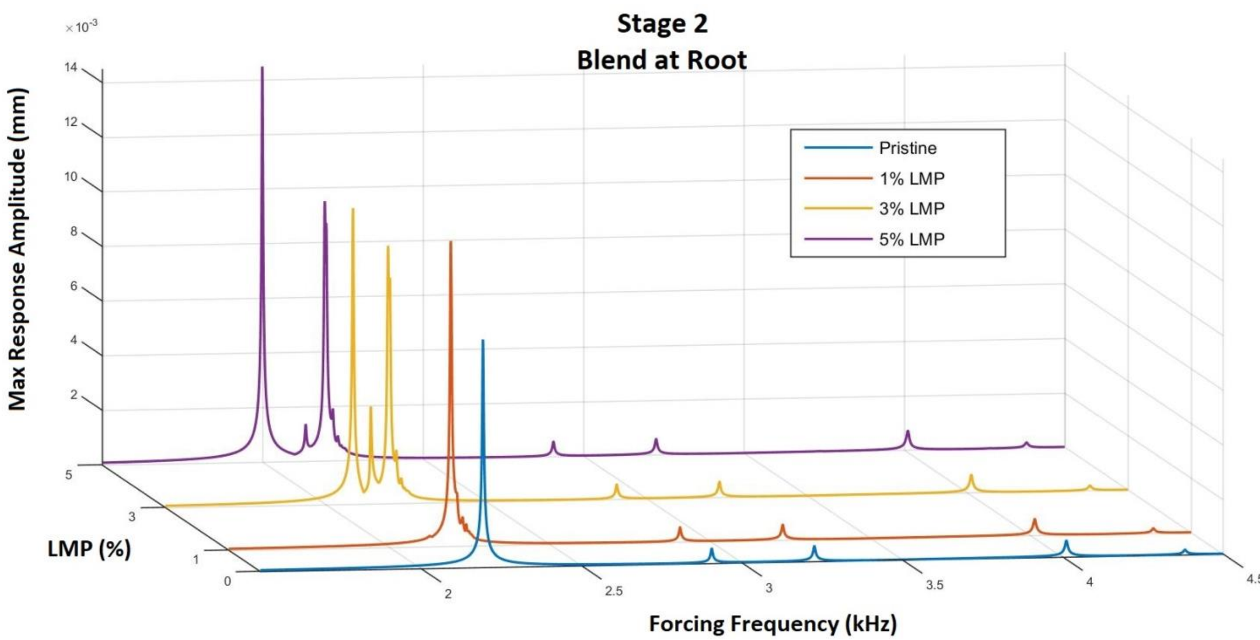


**Figure 19: Stage 2 Forced Responses for Blended Blade at the Root (1.5-4.5 kHz)**

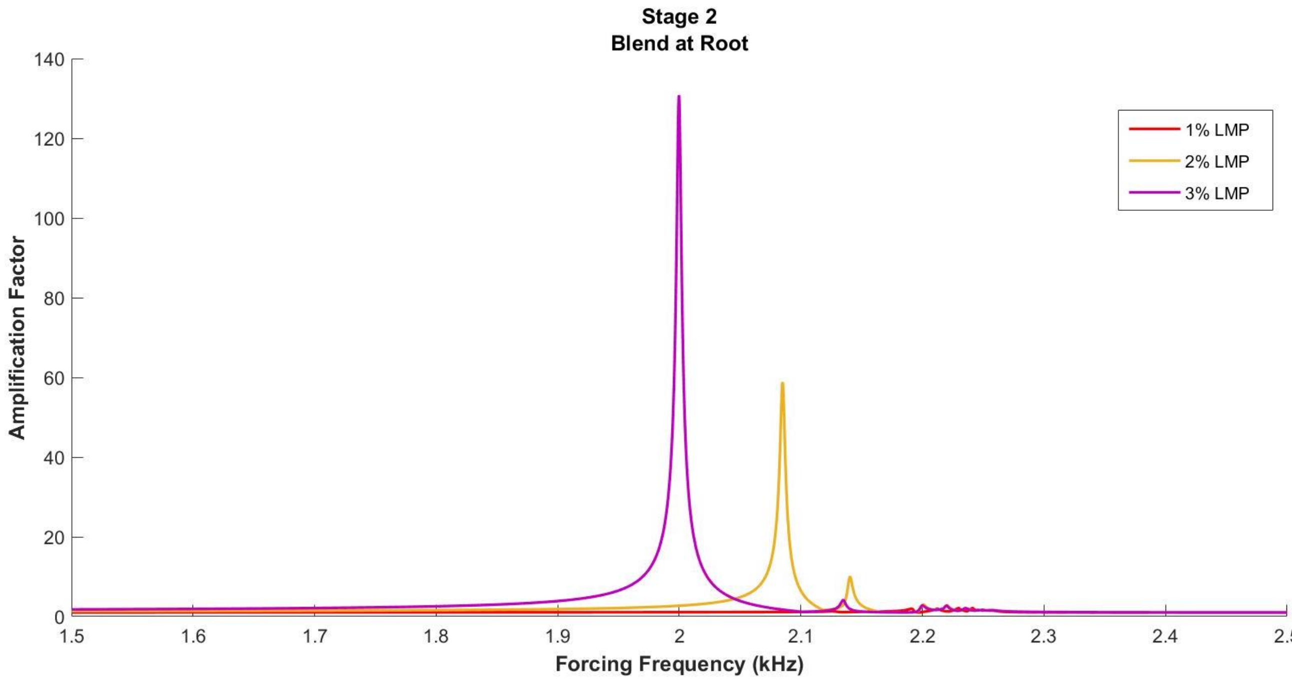


**Figure 20: Stage 2 Amplification Factor for Blended Blade at Root (1.5-4.5 kHz)**

The bent blades are also the most severe mistuning case in stage 1, although still relatively insignificant compared to stage 2. The root location's max amplitude is roughly twice that of the other two locations, but is not subjected to the peak splitting phenomenon seen in stage 2. It is the middle blend of 5% LMP that actually has the greatest peak deflection in stage 1 of 5.88%. Out of all large mistuning trials, this is the most dangerous for both stages and could have considerable impact on the structural performance of the system.

## 4.4 Large Mistuning Summary

An interim summary of large mistuning concludes that stage 2's response is generally much more amplified than stage 1 responses for all LMPs as expected. Blended blades show the most severe amplification followed by bends and then dents as negligible. Most of the increasing amplification trends are the same for both stages when LMP increases, but comparatively are greatly diminished in stage 1. The worst case scenario in stage 1 shows a peak amplitude increase of 5.88% compared to the 73.6% increase in stage 2 with blended blades. Blends are the most severe case particularly at the root and middle of the leading edge, by forming additional spikes in deflection at lower frequencies for stage 2. Ultimately, for a two-stage system, the only significant large mistuning damage that effects stage 1 is blends, as bent and dented blades have a relatively insignificant amplification.

# Chapter 5: Multistage Large and Small Mistuning Results

The results from large mistuning yield an understanding of how each type of large mistuning affects the multistage system. Now, small mistuning is added to create a more realistic system. The most severe case of large mistuning (5% LMP) is combined with a 100 randomly distributed small mistuning patterns with a 4% deviation in the material properties. A similar forced response analysis as shown in section 2.4 is used to find the average and maximum peak amplitude of all 100 tests. In addition, a small mistuning analysis with no large mistuning is conducted to compare the amplification of large mistuning.

## 5.1 Mixed Mistuning Stage 1 Results

The primary goal of this research is to understand the multistage effects of large and small mistuning that are now available with FCM-PRIME. Therefore we will first look at stage 1 that does not have any rogue blades, but is still affected by stage 2's large mistuning. It is expected that the mixed mistuning results should reflect the large mistuning results, and stage 1 should see less of a response than stage 2. In addition, based on chapter 4's large mistuning analysis, the blended blades should have a greater amplification than any other type of large mistuning.

The results for each type of mistuning have been summarized in figure 21. Each large mistuning case shows the peak response of 4 metrics: The pristine, with large mistuning only, the average peak amplitude of 100 mixed mistuning responses, and the maximum peak amplitude of those 100 mixed mistuning responses. Compared side by side, these metrics give the relative amplification of each mistuning parameter when compared to the pristine results.

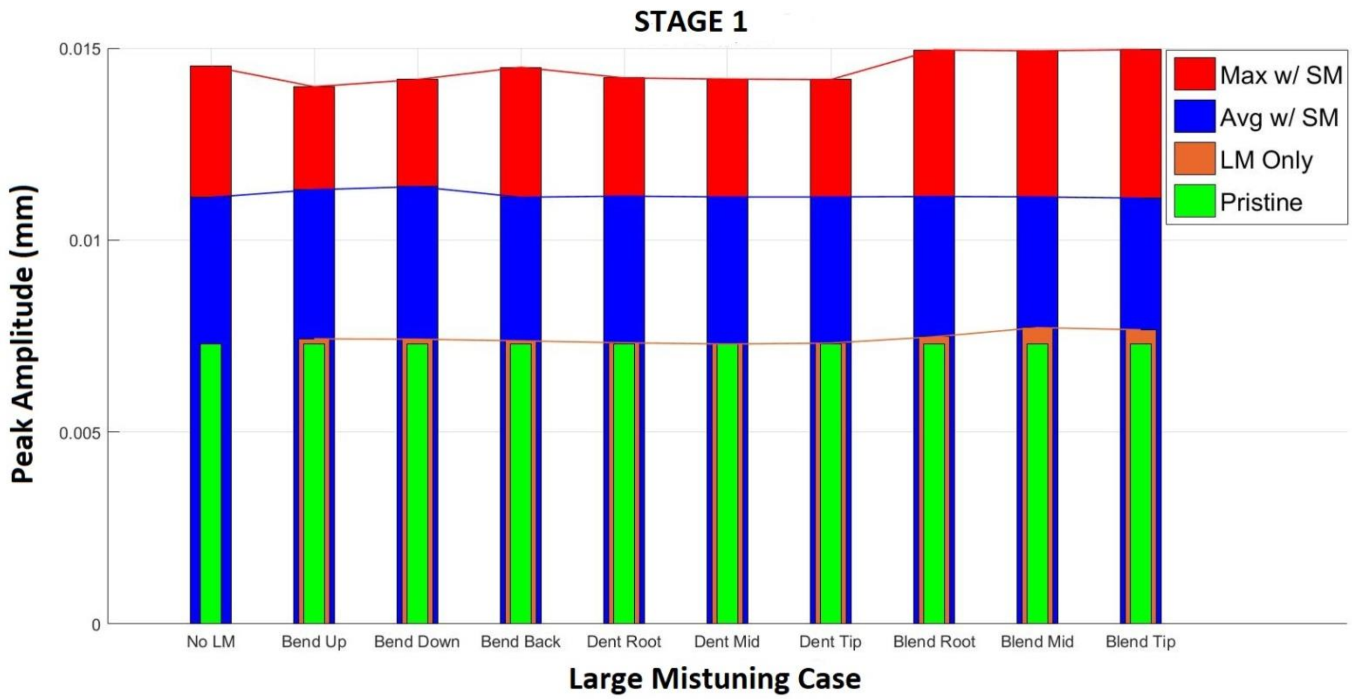


**Figure 21: Stage 1 Mixed Mistuning Results**

The first bar of figure 21shows the peak response when the system is subjected to small mistuning and no large mistuning. This is the baseline system that determines the amplification of the other mixed mistuning responses. With a 4% small mistuning, the average peak response is 0.0111 mm with a max peak response of 0.01453 mm. This gives the small mistuning only system a maximum peak amplification factor of 1.993 for stage 1. This demonstrates that small mistuning at 4% alone has a much more profound impact than the effects of large mistuning. This can be contributed to the fact that small mistuning is pervasive throughout both stages of the blade, while large mistuning is confined to stage 1.

However, as disproportionate as the effects of small mistuning are, it is still of interest to understand how small mistuning combines with large mistuning. For the case of bent and dented blades, the peak responses show insignificant deviation from the baseline response without any large mistuning (no LM bar). The dented blades consistently showed no change, while the upward and downward bends had their average response increase by 1.8% and 2.5%, respectively. However, in these two cases, as the average response increased, the maximum response decreased by 3.65% and 2.41%, respectively. The cause of this inverse relationship is not exactly known, but could possibly be contributed to a statistic outlier, and more than 100 mistuning patterns need to be run to confirm this trend.

The stage 1 results for blended blades also show some interesting behavior. As expected with the results from chapter 4, blended blades increase the maximum peak response by 2.89%, 2.75%, and 2.96% for the root, middle, and tip blends respectively. Unlike the bent blades, the average peak response with small mistuning remains constant with a change of less than 0.2% for all 3 blend locations. These results show that the increase in the large mistuning only peak responses, translate to the same trends as the maximum statistical response with small mistuning. The consistency in the average response and the increase in the max response seem to indicate that missing mass mistuning allows for a wider deviation in the statistical response, allowing for larger peak responses.

Another important conclusion can be made about the contribution of small mistuning with large mistuning. The small mistuning dampens the relative amplification of the large mistuning. Although the peak response still increases with the addition of small mistuning, the proportional contribution of large mistuning diminishes in stage 1. For example, let's examine the case of a blend to the middle of the leading edge. Without small mistuning, the rogue blade increases the peak response of the pristine system by 5.88% as shown in chapter 4. However the mixed mistuning results only show an increase of 2.75% in the peak response compared to a pristine system with small mistuning, and almost no change in the average response. Similar results follow for the other blend locations. It can be inferred that with significantly large mistuning cases like blends, the randomly distributed nature of small mistuning decreases the relative impact of the large mistuning on stage 1. It is also important to note that while the relative impact of large mistuning decreases, especially on the average, the potential for the worst case maximum response still increases.

The results ultimately show that mixed mistuning has a relatively small impact on stage 1 compared to the effects of small mistuning. For deformation mistuning such as dents and bends, the amplification is miniscule, and in some cases even reduces the maximum amplification. The effects of blends with small mistuning however have a noteworthy impact on stage 1, and should be considered.

## 5.2 Mixed Mistuning Stage 2 Results

The novelty of this research is in understanding the multistage response of mistuning, but the opportunity to examine the results of stage 2's behavior with mixed mistuning is also relevant. Since the rogue blade is applied to the stage 2, the results are more pronounced and offer some interesting insight. The stage 2 results come from the same 100 mixed mistuning forced response tests as stage 1, and still use the highest LMP of 5%. The results are also organized in similar fashion for easy comparison of mistuning.

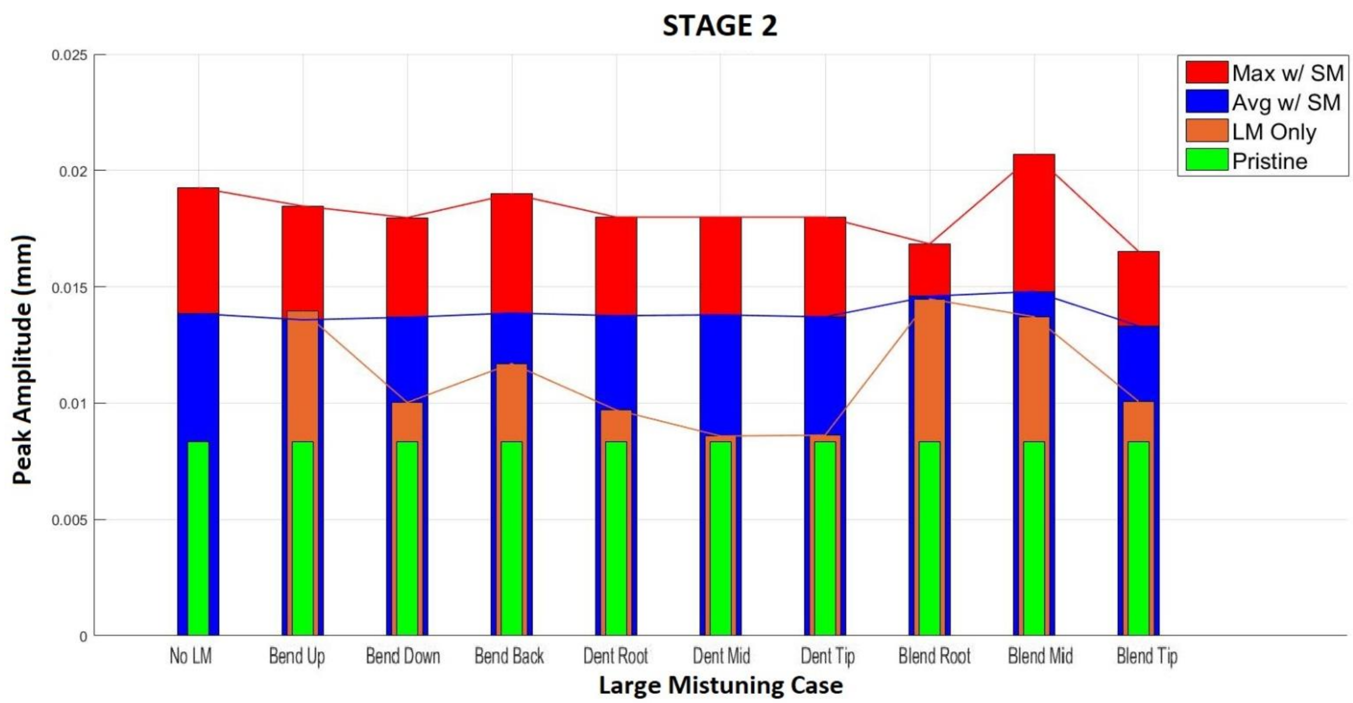


**Figure 22: Stage 2 Mixed Mistuning Results**

Figure 22shows the stage 2 peak response for mixed mistuning as well as small and large mistuning only. First examining the pristine case with small mistuning, we observe a large increase in the average and max peak amplitude of 65.9% and 130.8% respectively. This indicates that for the same level of small mistuning, stage 2 is much more responsive than stage 1. By comparison stage 1 had a maximum peak deflection increase of only 99.3%. This pristine case with small mistuning is used as the baseline for comparison when large mistuning is added.

The graph shows a surprising trend with the addition of large mistuning. In all cases except for the blend to the middle of the leading edge, the maximum response is actually lower than when there is no large mistuning. This is a counter intuitive result, as it is expected that a geometrically damaged blade would increase the response. In the most peculiar case of an upward bend, small mistuning can actually reduce the peak amplitude of the stage compared the large mistuning only case. Here the average peak amplitude with mistuning is 2.65% lower than without large mistuning. This is even more surprising by the fact that the upward bend is the most severe large mistuning only case of the bent blades. One possible reason for this phenomenon is that the small mistuning is distributing what would be localized energy from the damaged blade, thus decreasing the probability of a localized deflection in a single blade.

All three dent locations show near identical results with a 6.54% lower max peak deflection than the pure small mistuning cases. While the dent near the root has the greater amplification in the large mistuning case only, this unique behavior goes unobserved when small mistuning is included, as all dents have a uniform amplification affect.

Likewise to chapter 4's results, blended blades have the most volatile and unpredictable results. The most amplified blend location at the root has the greatest large mistuning only response, but the addition of small mistuning decreased the max response by 12.5% compared to no large mistuning. While the max amplitude decreases, the average response with small mistuning increases by 5.49%. This indicates small mistuning has the effect of a higher and tighter distribution of peak deflections for this case. In the other case of a blend at the middle of the leading edge, the average response increased with similar magnitude, but contrarily, the max response increased by 7.37%. Finally, the tip blend location is the only case to have a decrease in the average peak response of 3.76%. The tip blend also had the lowest maximum peak reduction of 14.2%.

With the erratic behavior of blends, the only general conclusion for this type of damage is that any amplification in the large mistuning only case correlates closely with the average response with small mistuning. However the maximum response fluctuates unpredictably as a function of blend location. Blends to the root of the leading edge have a significant reduction while blends to the middle have a significant increase in the max response when small mistuning is included. This means that missing mass damage is the most sensitive compared to deformation damage. This can be partly contributed to the significant amplification of lower frequencies as shown in chapter 4.

A general summary of stage 2 finds that mistuning amplification of the forced response is much greater than stage 1 as expected. It is observed that bends and dents do not significantly change the average peak amplification with the application of 100 small random small mistuning patterns. For these cases, the maximum amplification with small mistuning actually decreases. It is also found that blends have a much more erratic behavior for each location. The root location shows an increase in the average peak response, but also the largest decrease in the maximum response, while the middle location has the largest increase in the maximum response. Therefore is can be concluded that in stage 2, small mistuning has a major impact on the blends peak response. Blends are highly sensitive to their location, while deformation mistuning shows consistent decreases in the max amplitude with the inclusion of small mistuning.

# Chapter 6: Conclusion

The purpose of this research was to model the multistage behavior of a turbomachinery system damaged by small and large mistuning. This is done to understand how a variety of damaged blades in one stage affect other stages and compare the severity of the damage. In addition this work aims to demonstrate the capabilities of the FCM-PRIME method as a research tool to greatly reduce the amount of computational time needed to analyze the behavior of multistage mistuning.

## 6.1 Contributions

Mistuning is an inherent problem in turbomachinery that disrupts the ideal response turbine designers strive for, leading to structural fatigue and possibly failure. Computational modeling of the structural dynamics is one solution used to understand this vibration behavior. However, due to the extensive time required to model large systems, particularly in industry models, analysis is often limited to a single stage. This neglects the realist multistage effects that an actual turbine will experience.

The novelty of using FCM-PRIME is that multistage systems can now be modeled efficiently. Previously, the effects of small mistuning have been studied on a multistage system using similar reduced order models. This project seeks to add to that work by observing the effects of large mistuning, and the combination of large and small mistuning on a multistage system. In particular, this research is looking for any type of damage on one stage that could significantly affect the vibration response of another stage.

Engine manufacturers are increasing the use of blisks that are machined as one part, meaning that single damaged blades cannot be replaced. Overtime, mistuning from in service wear and manufacturing accrue on these turbine stages, but replacing the entire turbine stage is costly. It is important to understand what types of damage and severity is acceptable, and at what point these stages should be discarded. This research gives turbine engineers insight into how these types of damage are affecting other stages, and potentially compromising their structural behavior.

## 6.2 Additional Applications

This project utilizes the computational cost savings of the FCM-PRIME to analyze the effects of only a single large mistuned blade on a two stage system. However this ROM can be extended to model a variety of different parameters. A similar forced response analysis can be conducted for systems comprised of any number of stages. In addition, multiple different types of large mistuning can be applied to any number of blades at a time. These damaged blades can also be constructed with limitless possibilities, now the restraints of small mistuning only are removed. This project only tested a few cases and locations of the most common types of damage, to asses which types of damage are most severe from a high level. A project modifying any of the parameters can easily be conducted to expand on the mistuning knowledge gained in this project. These factors and any other additional work can be applied to industry with real turbines. There the computational models are orders of magnitude larger and greatly benefit from using FCM-PRIME. More specific analysis of a particular mistuning case, can be directly applicable to problems occurring in the field witnessed by engine manufacturers and maintenance teams.

## 6.3 Future Work

This work can be expanded on with the collection of more mistuning data to draw better conclusions. One possibility is to gain better resolution of the data by creating large mistuning models for large mistuning percentages of 2% and 4%. Conducting more small mistuning tests is also a possibly to ensure that the statistical data envelopes the whole distribution of possible max deflections. This ensures that the results are not due to the chance of random outliers. In addition, the process of creating and acquiring the prerequisite data for each large mistuning case can be streamlined. This efficiency would allow for faster modeling between computations, which would enable more the processing of more types of damage. Finally, an analysis of any of the other parameters stated in section 6.2 could branch off this research and contribute to a more complete understanding of mistuning.

## 6.4 Summary

Two major aspects of interest should be taken away from this research. First, the amplification of only large mistuning on the stage which it is applied to is much more severe than its neighboring stages, as expected. However, these effects should not be ignored in stage 1. The worst case of the blended blade has an increase of 5.88% in the peak deflection for stage 1, which is relatively low compared to the 73.6% increase in stage 2. In general without mistuning, dented blades have the least amplification, followed by bends, and then blended blades as the most severe. In addition it was also found that the amplification in both stages is highly sensitive to the direction of the bend and location of the blend. In particular blends show major amplification and growing of peaks at lower frequencies at the root and middle of the leading edge. Finally without mistuning it can be concluded that in most cases, the max amplification factor and peak response is proportional to the large mistuning percentage of the damaged blade.

The second major takeaway of this research is with respect to the inclusion of small mistuning. The primary multistage focus shows that large and small mistuning together have relatively little amplification on average peak response of stage 1 (compared to the pristine case with small mistuning). For bends and dents, there is actually a slight reduction in the maximum peak amplification. In contrast, the case of blended blades increases the max peak response by almost 3% for all locations. Meanwhile, stage 2 shows similar results with a decrease in the maximum amplitude for bends and dents. The response of blended blades in stage 2 is quite sensitive and volatile to location. The root location shows a major decrease in the max response (contrary to without small mistuning), while the middle location shows major increase in the max response. In summary, this research concludes that large mistuning has considerable amplification on stage 2 and lesser amplification in stage 1 that should not be ignored particularly for blended blades

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